Lecture 10: Lovász Local Lemma

- Let $\mathbb{A}_1, \ldots, \mathbb{A}_n$ be indicator variables for bad events in an experiment
- Suppose $\mathbb{P}[\mathbb{A}_i] \leqslant p$
- We want to avoid all the bad events
- If $\mathbb{P}[\neg\mathbb{A}_1 \ \wedge \ \cdots \ \wedge \ \neg\mathbb{A}_n] > 0$, then there exists a way to avoid all the bad events simultaneously
- Suppose, the event $\mathbb{A}_i$ is independent of all other events
- Then, it is easy to see that:

$$\mathbb{P}[\neg\mathbb{A}_1 \ \wedge \ \cdots \ \wedge \ \neg\mathbb{A}_n] \geqslant (1-p)^n > 0$$

- Lovász Local Lemma will help us conclude the same even in presence of "limited independence"

The Statement

Theorem (Lovász Local Lemma)

Let $(\mathbb{A}_1, \ldots, \mathbb{A}_n)$ *be a set of bad event. For each* $\mathbb{A}_i$*, where* $i \in [n]$*, we have* $\mathbb{P}[\mathbb{A}_i] \leqslant p$ *and each event* $\mathbb{A}_i$ *depends on at most* d *other bad events. If* $\mathrm{e}p(d+1) \leqslant 1$*, then*

$$\mathbb{P}[\neg\mathbb{A}_1 \ \wedge \ \cdots \ \wedge \ \neg\mathbb{A}_n] \geqslant \left(1 - \frac{1}{d+1}\right)^n > 0$$

The condition is also stated sometimes as $4pd \leqslant 1$, instead of $\mathrm{e}p(d+1) \leqslant 1$.

Application: k-SAT

- Let Φ be a k-SAT formula such that each variable occurs in at most $2^{k-2}/k$ different clauses
- Experiment: $\mathbb{X}_i$ be an independent uniform random variable that assigns the variable x_i a value from $\{\text{true}, \text{false}\}$
- Bad Event: For the j-th clause we have the bad event $\mathbb{A}_j$ that is the indicator variable for the bad event: The j-th clause is not satisfied
- Probability of Bad Event: For any j, note that

$$\mathbb{P}\left[\mathbb{A}_j\right] \leqslant \frac{1}{2^k},$$

Because there is at most one assignment of variables to make the clause false.

- Dependence: Note that the j-th clause has k literals, and each variable of the literal occurs in $2^{k-2}/k$ different clauses. So, the clause $\mathbb{A}_j$ can depend on at most $d = 2^{k-2}$ different bad events
- Conclusion: Note that $4pd = 1$, so Lovász Local Lemma implies that there exists an assignment that satisfies all the clauses in the formula simultaneously
- Observation: The probability p of each bad events does not depend on the overall problem instant size (i.e., the number of variables).

Application: Vertex Coloring

- Let G be a graph with degree at most Δ
- Experiment: $\mathbb{X}_v$ be the random variable that represents the color of the vertex v. Let $\mathbb{X}_v$ be a independent and uniformly random over the set $\{1, \ldots, C\}$
- Bad Event: For every edge e, we have a bad event $\mathbb{A}_e$ that is the indicator variable for both its vertices receiving identical color
- Probability of the Bad Event: Note that $\mathbb{P}[\mathbb{A}_e] = \frac{1}{C}$
- Dependence: Note that the event $\mathbb{A}_e$ does not depend on any other event $\mathbb{A}_{e'}$ if the edges do not share a vertex. So, the event $\mathbb{A}_e$ depends on at most $2(\Delta - 1)$ other bad events
- Conclusion: A valid coloring exists if $4pd \leqslant 1$, i.e., $C \geqslant 8(\Delta - 1)$

Application: Vertex Coloring (Bad Bound)

- Let G be a graph with degree at most Δ
- Experiment: $\mathbb{X}_v$ be the random variable that represents the color of the vertex v. Let $\mathbb{X}_v$ be a independent and uniformly random over the set $\{1, \dots, C\}$
- Bad Event: For every edge v, we have a bad event $\mathbb{A}_v$ that is the indicator variable for one of the neighbors of v receiving the same color as v
- Probability of the Bad Event: Note that $\mathbb{P}[\mathbb{A}_v] \leqslant 1 - \left(1 - \frac{1}{C}\right)^{\Delta}$
- Dependence: Note that the event $\mathbb{A}_v$ does not depend on any other event $\mathbb{A}_{v'}$ if $\{v\} \cup N(v)$ does not intersect with $\{v'\} \cup N(v')$. So, the event $\mathbb{A}_e$ depends on at most $\Delta + \Delta(\Delta - 1) = \Delta^2$ other bad events
- Conclusion: A valid coloring exists if $4pd \leqslant 1$, i.e., $C \geqslant$???

Proof of Lovász Local Lemma

Claim

Let $S \subseteq \{1, \ldots, n\}$, then we have:

$$\mathbb{P}\left[\mathbb{A}_i \,\middle|\, \bigwedge_{k \in S} \neg\mathbb{A}_k\right] \leqslant \frac{1}{d+1}$$

Assuming this claim, it is easy to prove the Lovász Local Lemma.

$$\mathbb{P}\left[\bigwedge_{i=1}^{n} \neg\mathbb{A}_i\right] = \prod_{i=1}^{n} \mathbb{P}\left[\neg\mathbb{A}_i \,\middle|\, \bigwedge_{k<i} \neg\mathbb{A}_k\right]$$

$$\geqslant \prod_{i=1}^{n}\left(1 - \frac{1}{d+1}\right) = \left(1 - \frac{1}{d+1}\right)^n > 0$$

Proof of the Claim

- We will proceed by induction on $|S|$
- Base Case: If $|S| = 0$, then the claim holds, because:

$$\mathbb{P}\left[\mathbb{A}_i \;\middle|\; \bigwedge_{k \in S} \neg \mathbb{A}_k\right] = \mathbb{P}[\mathbb{A}_i] \leqslant p \leqslant \frac{1}{e(d+1)} \leqslant \frac{1}{d+1}$$

- Assume that for all S $|S| < t$, the claim holds
- We will prove the claim for $|S| = t$. Suppose D_i be the set of all j such that the bad event $\mathbb{A}_i$ depends on the bad event $\mathbb{A}_j$
- **Easy Case.** Suppose $S \cap D_i = \emptyset$. This case is easy, because

$$\mathbb{P}\left[\mathbb{A}_i \;\middle|\; \bigwedge_{k \in S} \neg \mathbb{A}_k\right] = \mathbb{P}[\mathbb{A}_i] \leqslant p \leqslant \frac{1}{e(d+1)} \leqslant \frac{1}{d+1}$$

Proof of the Claim II

- **Remaining Case.** Suppose $S \cap D_i \neq \emptyset$.

$$
\begin{aligned}
\mathbb{P}\left[\mathbb{A}_i \,\middle|\, \bigwedge_{k\in S} \neg\mathbb{A}_k\right] &= \mathbb{P}\left[\mathbb{A}_i \,\middle|\, \bigwedge_{k\in D_i} \neg\mathbb{A}_k, \bigwedge_{k\in S\setminus D_i} \neg\mathbb{A}_k\right] \\
&= \frac{\mathbb{P}\left[\mathbb{A}_i, \bigwedge_{k\in D_i} \neg\mathbb{A}_k \,\middle|\, \bigwedge_{k\in S\setminus D_i} \neg\mathbb{A}_k\right]}{\mathbb{P}\left[\bigwedge_{k\in D_i} \neg\mathbb{A}_k \,\middle|\, \bigwedge_{k\in S\setminus D_i} \neg\mathbb{A}_k\right]} \\
&\leqslant \frac{\mathbb{P}\left[\mathbb{A}_i \,\middle|\, \bigwedge_{k\in S\setminus D_i} \neg\mathbb{A}_k\right]}{\mathbb{P}\left[\bigwedge_{k\in D_i} \neg\mathbb{A}_k \,\middle|\, \bigwedge_{k\in S\setminus D_i} \neg\mathbb{A}_k\right]} \\
&= \frac{\mathbb{P}\left[\mathbb{A}_i\right]}{\mathbb{P}\left[\bigwedge_{k\in D_i} \neg\mathbb{A}_k \,\middle|\, \bigwedge_{k\in S\setminus D_i} \neg\mathbb{A}_k\right]}
\end{aligned}
$$

Proof of the Claim III

- Suppose $D_i = \{i_1, \ldots, i_z\}$
- Using chain rule, we can write the denominator

$$\mathbb{P}\left[\bigwedge_{k \in D_i} \neg \mathbb{A}_k \;\middle|\; \bigwedge_{k \in S \setminus D_i} \neg \mathbb{A}_k\right]$$

as follows

$$\prod_{\ell=1}^{z} \mathbb{P}\left[\neg \mathbb{A}_{i_\ell} \;\middle|\; \bigwedge_{k \in S \setminus D_i} \neg \mathbb{A}_k, \bigwedge_{k' \in \{i_1, \ldots, i_{\ell-1}\}} \neg \mathbb{A}_{k'}\right]$$

- Note that each probability term is condition on $< t$ bad events. So, we can apply the induction hypothesis. We get

$$\mathbb{P}\left[\bigwedge_{k\in D_i} \neg\mathbb{A}_k \,\middle|\, \bigwedge_{k\in S\setminus D_i} \neg\mathbb{A}_k\right] \geqslant \prod_{\ell=1}^{z}\left(1-\frac{1}{d+1}\right)$$
$$= \left(1-\frac{1}{d+1}\right)^z \geqslant \left(1-\frac{1}{d+1}\right)^d$$
$$\geqslant \frac{1}{\mathrm{e}}$$

- Now, let us return to our original expression

$$\mathbb{P}\left[\mathbb{A}_i \,\middle|\, \bigwedge_{k\in S} \neg\mathbb{A}_k\right] \leqslant \frac{\mathbb{P}\left[\mathbb{A}_i\right]}{\mathbb{P}\left[\bigwedge_{k\in D_i}\neg\mathbb{A}_k \,\middle|\, \bigwedge_{k\in S\setminus D_i}\neg\mathbb{A}_k\right]}$$
$$\leqslant \mathrm{e}\mathbb{P}\left[\mathbb{A}_i\right] \leqslant \frac{1}{d+1}$$

Proof of the Claim V

- This completes the proof by induction
- We will prove a more general result in the next lecture